J. Fluid Mech. (1972), *vol.* 51, *part* 3, *pp.* 613–618
Printed in Great Britain

Wake collapse in a stratified fluid: linear treatment

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(Received 12 August 1971)

The linear initial-value problem of a partially mixed cylindrical wake in a uniformly stratified fluid is formulated and exact solutions are given for the density and velocity fields inside and just outside the original cylinder. An asymptotic expression for the far-field internal wave radiation is given and the corresponding solutions for a spherical wake geometry are noted. The treatment emphasizes the inadequancy of the usual linear Boussinesq approximation to describe the detailed nature of similar problems, in particular the fully mixed wake-collapse problem.

1. Introduction

In recent years a number of authors have considered the problem of wake collapse in an incompressible stratified fluid. The theoretical treatments (Mei 1969; Miles 1971) have usually involved a version of the linear Boussinesq approximation, though some numerical work has appeared. While there seems to be a consensus about the phenomenology in the case of a fully mixed wake (Mei 1969; Schooley & Stewart 1963; Wu 1965), for which the linear approximation is of course invalid, it has seemed to us useful to treat a particularly simple version of the linear problem, for which an exact solution is obtainable. The results contain some surprises and provide some insight into the limitations of the linear treatment.

2. Basic equations

We study a linearly stratified fluid, with no boundaries, whose unperturbed density $\rho_0(z)$ in the vicinity of the region of interest varies only in the vertical, z, direction and does so gradually enough to justify a linear approximation. Thus

$$\rho_0(z) \approx \rho_{00} - \beta z, \tag{1}$$

where ρ_{00} is a 'mean' density for the problem. The initial-value problem is obtained by perturbing the density slope inside a cylinder of radius a;

$$\delta\rho(t=0) = \begin{cases} \epsilon z & (r < a), \\ 0 & (r > a), \end{cases} \tag{2}$$

where the fully mixed case can be obtained formally be setting $\epsilon = \beta$. This state of affairs is illustrated in figure 1. We are working in a two-dimensional coordinate system with axes z (vertical) and x (horizontal), with everything assumed independent of y. We shall also use polar co-ordinates (r, α) in the plane of the problem, with the polar axis vertical.

Newton's second law, in linearized form, is

$$\rho(\partial \mathbf{v}/\partial t) = -\nabla p - \rho g \hat{\mathbf{z}}, \tag{3}$$

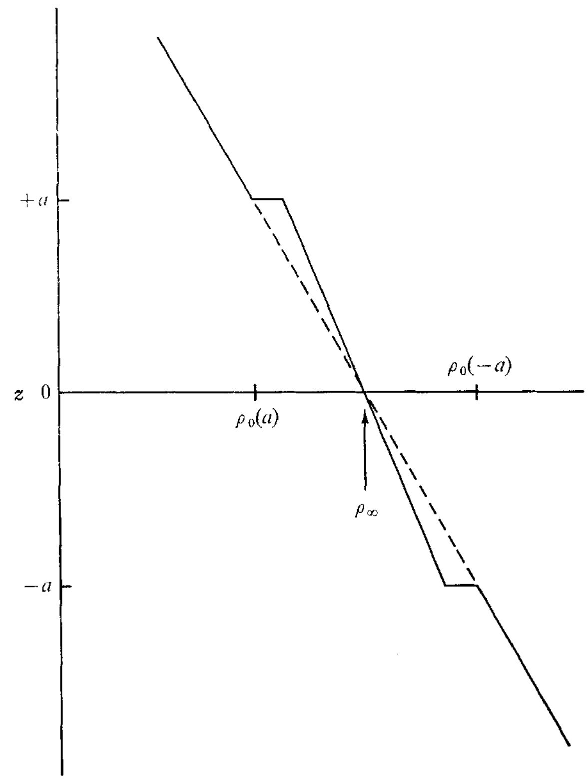

FIGURE 1. Initial density profile $\rho_0(z)$ at $x = 0$.

where $\mathbf{v}$ is the fluid velocity (in the x, z plane), p is the pressure, g the acceleration of gravity and $\hat{\mathbf{z}}$ a vertical unit vector. Since the fluid is assumed incompressible we have

$$\rho\nabla . \mathbf{v} = \partial\rho/\partial t + (\mathbf{v} . \nabla)\rho = 0, \tag{4}$$

which allows us to introduce the stream function $\boldsymbol{\psi}$, a vector in the y direction defined by

$$\mathbf{v} = \nabla \times \boldsymbol{\psi}. \tag{5}$$

The curl of (3) yields
$$\rho_{00} \frac{\partial}{\partial t} \nabla \times (\nabla \times \boldsymbol{\psi}) = -g\nabla\rho \times \hat{\mathbf{z}}, \tag{6}$$

where the insertion of ρ_{00} is usually called the Boussinesq approximation. It amounts to taking into account only density gradients that lead to a force (when multiplied by g). Differentiating (6) with respect to time leads, in view of (4), to

$$\frac{\partial^2}{\partial t^2} \nabla^2 \boldsymbol{\psi} = -N^2 \frac{\partial^2 \boldsymbol{\psi}}{\partial x^2}, \tag{7}$$

where we have used the linear approximation and defined the Väisälä frequency N by

$$N^2 \equiv -(g/\rho_0)(d\rho_0/dz) \approx g\beta/\rho_{00}, \tag{8}$$

which we will treat as a constant hereafter. (Obviously, we could have chosen ρ_0 to be exponential instead of as given in (1).)

Clearly, in view of (4), $\delta\rho$ can be obtained from ψ, so that (7), with the initial condition (2), is our problem. It begs for Fourier analysis, and the (exact) answer is

$$\delta\rho = \epsilon z a^2 \int_0^\infty k\,dk\, J_2(ka) \frac{J_1[(k^2z^2+(Nt\pm kx)^2)^{\frac{1}{2}}]}{(k^2z^2+(Nt\pm kx)^2)^{\frac{1}{2}}}, \tag{9}$$

where the $\pm$ means that the integral should be evaluated with each of the two signs and the results *averaged*; J_n is the nth order Bessel function of the first kind.† The corresponding expression for ψ is obtained from (9) by multiplying the integrand by $2N/k\beta$ and taking the *difference* of integrals for the two signs. (There are other simpler ways to get the velocity distribution.) It is worth re-emphasizing that (9) is the exact solution to a *linear* problem.

3. Quadratures

The evaluation of (9) is greatly facilitated by the observation that the expression in the integrand involving the square roots admits of an expansion in terms of Chebyshev polynomials of the second kind U_m. Explicitly, if

$$w^2 = u^2+v^2-2uv\cos\gamma$$

then

$$J_1(w)/w = 2\sum_{m=0}^{\infty}(1+m)\frac{J_{1+m}(u)}{u}\frac{J_{1+m}(v)}{v}U_m(\cos\gamma), \tag{10}$$

(see HMF, equation 9.1.80) where

$$U_m(\cos\gamma) = \sin((m+1)\gamma)/\sin\gamma. \tag{11}$$

Thus, after a bit of arithmetic and after replacing γ by $\alpha-\frac{1}{2}\pi$, we find that

$$\delta\rho = \frac{2\epsilon z a^2}{r}\sum_{l=0}^{\infty}(-1)^l(2l+1)\frac{\cos[(2l+1)\alpha]}{\cos\alpha}\frac{J_{2l+1}(Nt)}{Nt}\int_0^\infty J_2(ka)\,J_{2l+1}(kr)\,dk. \tag{12}$$

Before proceeding to the final form some observations about (12) are in order.

First and foremost, inside the original cylinder, where $r < a$, only the first term in (12) is non-vanishing. Thus, exactly,

$$\delta\rho = 2\epsilon z J_1(Nt)/Nt \quad \text{for} \quad r < a. \tag{13}$$

The original linear (proportional to z) perturbation inside the cylinder remains linear, overshoots its correction and finally damps out.‡

† Here, and henceforth, we use the notation of Abramowitz & Stegun (1970), whose book *Handbook of Mathematical Functions* is hereafter referred to as HMF.

‡ It is worthwhile noting that the corresponding problem with a *spherical* wake geometry also admits a closed solution for $r < a$. We find $\delta\rho(\mathbf{r},t) = \frac{3}{2}\pi\epsilon z E_2(Nt)/Nt$ for $r < a$, where $E_2(Nt)$ is the second-order Weber function. The similarity to the problem under investigation is evident, the behaviour of $E_2(Nt)$ being qualitatively the same as $J_1(Nt)$.

The fluid velocities for $r < a$ are most easily obtained from (13) by noting that $v_z = (1/\beta)\,\partial(\delta\rho)/\partial t$, so that

$$v_z = -(2\epsilon z/\beta t)\,J_2(Nt). \tag{14a}$$

In view of the incompressibility condition (4) we also have

$$v_x = (2\epsilon x/\beta t)\,J_2(Nt), \tag{14b}$$

so that the fluid particles inside the original cylinder move on right hyperbolae, overshooting their ultimate positions on the first pass. The ultimate displacement of a fluid particle that starts inside the original cylinder at (x, z) is given by

$$\Delta x = \epsilon x/\beta, \quad \Delta z = -\epsilon z/\beta, \tag{15}$$

so that the fluid particle finally moves just far enough down its own hyperbola to reach the level which is appropriate for it. In particular, the original circular cross-section of the cylinder deforms into an ellipse of semi-major axis $a(1+\epsilon/\beta)$ and semi-minor axis $a(1-\epsilon/\beta)$ after oscillating around this shape.

We now turn to the behaviour of the fluid for $r > a$, to which all terms in (12) *except* the first contribute. For $r > a$ and $l \geqslant 1$ we need

$$\int_0^\infty J_2(ka)\,J_{2l+1}(kr)\,dk = \frac{a^2}{r^3}P_{l-1}^{(2,0)}\left(1-\frac{2a^2}{r^2}\right), \tag{16}$$

(see HMF, equations 11.4.34 and 15.4.6) where $P_n^{(\alpha,\beta)}(\xi)$ are the Jacobi polynomials, which for this case can be reduced to a more recognizable form involving the familiar Legendre polynomials P_l:

$$P_{l-1}^{(2,0)}(\xi) = \frac{r^4}{4a^4}\left[P_{-1}(\xi) - 2P_l(\xi) + P_{l+1}(\xi) + \frac{P_{l-1}(\xi)-P_{l+1}(\xi)}{2l+1}\right]. \tag{17}$$

We have then, for $r > a$:

$$\delta\rho(\mathbf{x}, t) = 2\epsilon z\frac{a^4}{r^4}\sum_{l=1}^{\infty}(-1)^l(2l+1)\frac{\cos[(2l+1)\,\alpha]}{\cos\alpha}\frac{J_{2l+1}(Nt)}{Nt}P_{l-1}^{(2,0)}(\xi), \tag{18}$$

where we have introduced $\xi = 1-2a^2/r^2$. As

$$r \to a^+, \quad \xi \to -1 \quad \text{and} \quad P_{l-1}^{(2,0)}(-1) = (-1)^{l-1} \tag{19}$$

so

$$\delta\rho(r\to a^+) = -2\epsilon z\sum_{l=1}^{\infty}\frac{\cos[(2l+1)\,\alpha]}{\cos\alpha}\frac{J_{2l+1}(Nt)}{Nt}(2l+1). \tag{20}$$

Comparing this with (13), we find an exact expression for the density discontinuity across the surface of the cylinder

$$\begin{aligned}\Delta\rho &\equiv \delta\rho(r\to a^-) - \delta\rho(r\to a^+)\\ &= 2\epsilon z\sum_{l=0}^{\infty}(2l+1)\frac{\cos[(2l+1)\,\alpha]}{\cos\alpha}\frac{J_{2l+1}(Nt)}{Nt}.\end{aligned} \tag{21}$$

This sum can be extended to $-\infty$ and carried out exactly. We then find

$$\Delta\rho = \epsilon z\,\cos(Nt\sin\alpha), \tag{22}$$

an unexpected result. From (22) and (13) we find

$$\delta\rho(r\to a^+) = \epsilon z[2\,J_1(Nt)/Nt - \cos(Nt\sin\alpha)], \tag{23}$$

which does *not* go to zero as $t \to \infty$ but oscillates more and more rapidly as a function of α. Such behaviour is indicative of an instability in a more realistic calculation, though the relation to the expected Kelvin–Helmholtz instability on the surface is unclear.

We turn finally to the behaviour at large distances, for which we need the behaviour of the $P_l^{(2,0)}(\xi)$ for ξ near unity:

$$P_l^{(2,0)}(\xi) \to (r^2/a^2)\, J_2(2l\,a/r) \quad (r \gg a), \tag{24}$$

in which we have also assumed $1 \ll l \ll r^2/a^2$ for convenience. It will be seen below that this is the range of interest for l. This leads to

$$\delta\rho \to \frac{2\epsilon z a^2}{r^2} \sum_{l=1}^{\infty} (-1)^l (2l+1) \frac{\cos[(2l+1)\alpha]}{\cos\alpha} \frac{J_{2l+1}(Nt)}{Nt} J_2\left[\left(2l\frac{a}{r}\right)\right]. \tag{25}$$

(Recall, as always, that v_z can be obtained from $\delta\rho$ through $v_z = (1/\beta)\,\partial(\delta\rho)/\partial t$, and v_x from v_z through the incompressibility condition (4).)

For any given point (r, α) at any time, the terms in the sum (25) increase in magnitude with l and oscillate rapidly in sign, so it is appropriate to look for a 'constant-phase' value of l as a means of estimating the sum. By either this method, or by a direct saddle-point integration in (9), we find that

$$\delta\rho \to \frac{2\epsilon z a^2}{r^2} \tan\alpha \sin(Nt\cos\alpha)\, J_2\left(\frac{Nta}{r}\sin\alpha\right) \tag{26}$$

for $r \gg a$. This represents a pulse of frequency $N\cos\alpha$ which passes a point (r, α) at a time $t \sim r/Na\sin\alpha$. The denominator is the group velocity of a wave of wavenumber $\sim 1/a$, in the correct direction, where $N\cos\alpha$ is the frequency of such a wave. (Recall that the group and phase velocities of Väisälä waves are mutually orthogonal.)

4. Conclusions and caveats

The behaviour of the solution (26) for large distances from the source contains no surprises and represents the radiation from the source of the expected pulse of Väisälä waves, necessary to get rid of the energy stored in the initial perturbation.

On the other hand, the behaviour near the perturbation has a number of unrealistic features. The rapidly oscillating (in space) behaviour just outside the original cylinder at long times cannot appear in the solution of a realistic hydrodynamic problem. Since the solution is exact, this difficulty must be ascribed to the model. In addition, the velocity of the fluid displays an infinite shear on the surface of the cylinder, and this too would lead to a Kelvin–Helmholtz instability in a realistic problem. It would seem foolhardy, therefore, to set $\epsilon = \beta$ and use a linear treatment to study the fully mixed wake-collapse problem.

This research was supported in part by the National Science Foundation.

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